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Précis for A Sense of Proportion: Using perception to ground number symbols

NOTE: *The colloquium talk that I will be presenting this week was not based upon an existing manuscript. Below, I provide a brief synopsis of the argument that I will present. Much of the talk will be dedicated to providing theoretical arguments and experimental data to corroborate these claims. Following the synopsis of my argument below, I also include some bits of background information that can both help make the talk easier to process and provide some citations for those interested in reading further about the processes that I will discuss. I apologize in advance for the somewhat fragmented nature of what follows.*

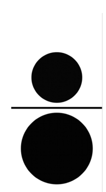
Basic mathematical competence is essential for participation in modern society. Indeed, mathematical competence is an important determinant of children's later educational and occupational prospects (Kilpatrick, Swafford & Findell, 2001). Several converging lines of research suggest that acquisition of a robust sense of number is an essential first step on the road to mathematical competence (Ansari, 2008; Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 2009); Gallistel & Gelman, 2000; Halberda & Feigenson, 2008; Landerl, Bevan & Butterworth, 2004; National Math Panel, 2008). Accordingly, it is of fundamental importance that we answer the question of how numerical symbols come to be meaningful in the first place. That is, how is it that symbolic numbers – relatively recent inventions on the evolutionary time scale – come to be associated with conceptions of specific magnitudes? How do we get a sense for what symbolic numbers mean?

Often, we resort to using collections of countable objects to pair number symbols with specific magnitudes. We begin to teach children about number by counting, accompanied by pointing out objects in a counted set in succession. This practice is so widespread that we have taken it to be that *natural* path to teaching and learning about number. The privileged position of natural numbers is epitomized by Kronecker's oft-quoted charge that, "God created the natural numbers; all else is the work of man."[1]

In this talk, I will present a second and complementary option for grounding number symbols – one that rests on our perceptual abilities to judge (and to match) proportions between continuous stimuli of various sizes. The implication is that nature may have provided us with a cognitive architecture that allows perceptual access to more than just natural numbers. I will provide clear evidence that this "sense" of proportion exists and discuss the potential applications for pedagogy.

I. THE ARGUMENT – PROPORTION MATCHING CAN BE USED TO MAKE NUMBERS MEANINGFUL.

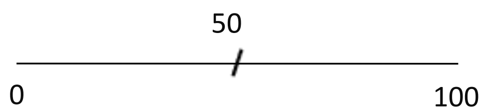
A. Human beings have a perceptually based ability to judge the proportions between magnitudes that are presented in the same representational format. For example, people can sense that the proportion between the two circles presented here is about ½ (I'll prove it):

a. 

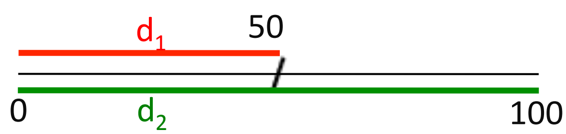
B. Human beings can match these proportions across representational formats. It's actually required when we do number line estimation tasks. In order to properly place the hatch

[1] Sometimes the quote is translated as referring to integers instead of to natural numbers.

mark on a task that asks you to estimate where 50 should go on a number line that goes from 0 to 100, you have to use this ability to match proportions:

a. 

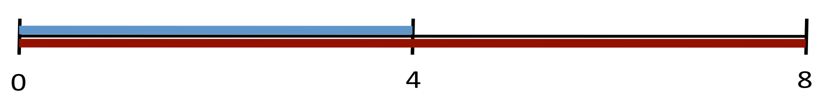
b. The only way to complete this task is to match the appropriate proportion of two line lengths (the length from 0 to 50 and the total length) to the proportion between the stimulus and the rightmost anchor of the number line (50:100).

c. 

so that

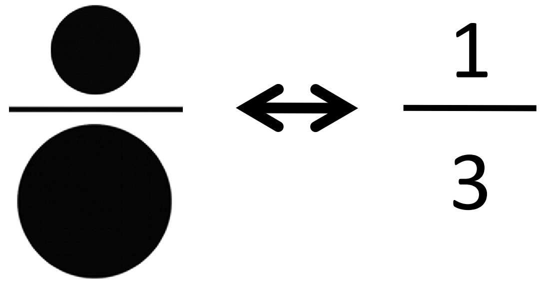
$$\frac{d_1}{d_2} = \frac{50}{100}$$

C. Because these tasks a) are inherently relational, and b) involve matching perceptually perceived proportions to symbolic proportions, they might be used to bootstrap existing symbolic knowledge via a sort of perceptual analogy. For instance, the setup below should be helpful for children who understand the symbol "4" but have not yet mastered the symbol "8". It rests on our perceptual abilities to match proportions. It shows that 8 is twice as big as 4.

a. 

D. If we can perceptually access the ratios between different magnitudes, on some level, we may have perceptual access to rational (even real) numbers. It has yet to be shown whether or not we can harness this system for practical use, but one thing is certain: in the figure below, there is a 1-to-1 map between the visuospatial proportion presented and the fraction 1/3.

a.



I will present some evidence that suggests that we do have a *sense* of proportion, and that it is actually a perceptual phenomenon that is substantially different from our standard count-based sense of fractions which is predicated on equisection of the whole into discrete units.

E. When you strip this argument to bare bones, it proposes that we can use continuous magnitudes – things that are inherently non-numerical – to gain access to numbers. The key is to use them relationally to facilitate 1-to-1 maps to numbers. Because this ability to judge and match proportions is innate to our cognitive architecture, it is a tool that we should attempt to exploit for pedagogical purposes. It may be a large factor underlying recent success using number lines as pedagogical tools. Explicit attempts to leverage this system may pay dividends both with typically and atypically developing populations.

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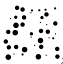
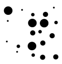
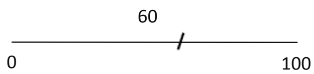
II. THE NUMBER SENSE(S)

The term 'number sense' is used by diverse camps, sometimes to mean very different things (Berch, 2005). For the purposes of introducing the series of experiments that I will discuss in the colloquium, this section will focus on two conceptions of 'number sense' which are thought to contribute to early understanding of numerical magnitude: 1) the approximate number system (ANS) that allows us to discriminate between nonsymbolic numerosities (e.g.  *, and*  *2) the ability to generate estimates of numerical magnitude using number lines (e.g.* 

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Sense 1 – The Approximate Number System (ANS)

Several strands of converging evidence have led some to suggest that meaningful understanding of the magnitude of symbolic number is rooted in the evolutionarily ancient ANS (e.g. Nieder & Dehaene, 2009; Mazzocco & Devlin, 2008; Nieder & Dehaene, 2009). The ANS is a perceptual system that allows animals to discriminate between two sets of nonsymbolic numerosities. Notably, the system is dependent upon the ratios between the sets as opposed to the raw numerical differences between them. Hence, it is easier to discriminate between arrays of 40 and 80 dots (a 1:2 ratio) than it is to discriminate between arrays of 120 and 160 dots (a 3:4 ratio). This system is 1) found across a wide range of species and 2) present in humans at birth, although the acuity of it is enhanced with development (Dehaene, 1997; Halberda & Feigenson, 2008; Hubbard, Diester, Cantlon, Ansari, van Opstal & Troiani, 2008). Importantly, this ability to discriminate between nonsymbolic number seems to work in much the same way as our abilities to discriminate between magnitudes more generally[2] (Moyer & Landauer, 1967).

This ratio dependent discrimination system also leads to some characteristic patterns on

[2] Here, I mean our abilities to discriminate between pairs of magnitudes in many different domains: temporal durations, lengths, masses, areas of circles, etc. See Feigenson, 2007 for a developmental perspective. Stevens, 1970 is a (slightly less accessible, but equally brief) intro to the concept for adult participants.

estimation tasks. Simply put, perceived differences in magnitude are somewhat distorted relative to actual differences in magnitude because our perceptual system is sensitive to ratios. For example, when presented with dot arrays, the difference between 10 and 20 dots is perceived as the same as the difference between 20 and 40 dots or the difference between 40 and 80 dots, because the ratios between the pairs is constant.[3] As a result, when people are asked to estimate how many dots are in various arrays, presented one at a time, plotting the estimates vs. the actual numbers of dots in the arrays leads to a function like the one presented below (Indow & Ida, 1977; Izard & Dehaene, 2008; Taves, 1941).

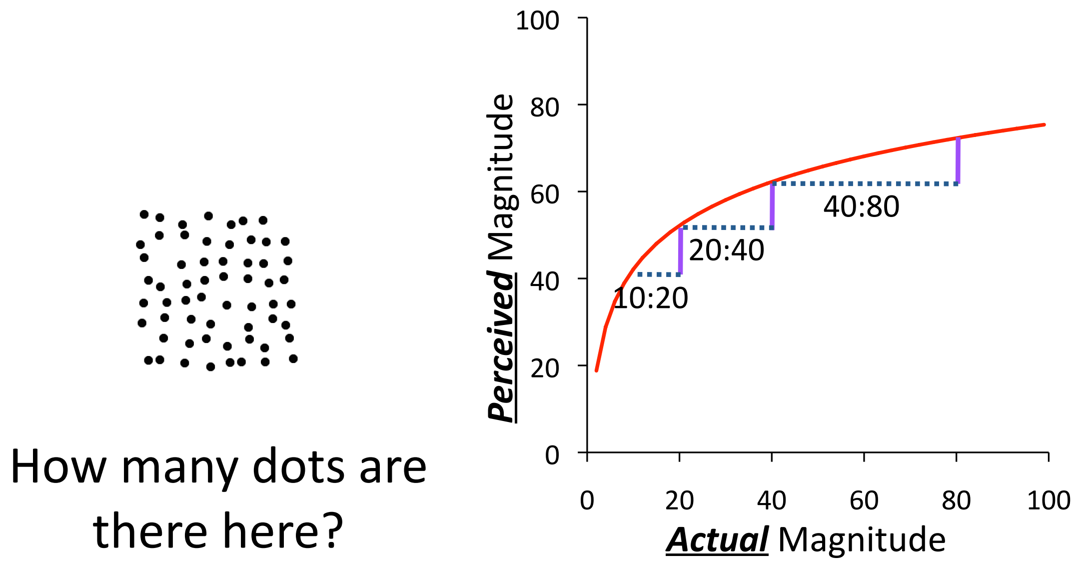

FIGURE: When estimating how many dots are in various individual arrays like the one presented on the left, graphs of median estimates vs. the actual stimulus number usually look like the graph on the right.

Past research has shown that our abilities to discriminate between pairs of symbolic numbers follow similar rules. The seminal work by Moyer and Landauer (1967) is perhaps the most well known of these studies. In that experiment, the authors presented adult subjects with

[3] This is provided that a person is required to answer quickly and without counting – an estimation at a glance.

pairs of single digit numerals from 1-9 and instructed them to indicate which of the two digits was larger. Participants were told to respond as quickly as possible without making errors. Analysis of reaction time data showed that the ability to discriminate between these symbolic numbers was also dependent on the ratios between the numerical stimuli presented. The closer the ratio was to 1:1, the longer the participants took to react.

This research was an anchor for other research that suggests that we use the ANS to ground symbolic numbers. Simple intuition suggests that our sense of the meaning of symbolic numbers should be rooted in displays of discrete quantity: we begin to teach children about number by counting, often accompanied by pointing out objects in a counted set in succession. With this experiment, however, there was clear evidence that residues of the very ways that we perceive of discrete sets of objects (qua perceptual magnitudes) were present in the ways that we processed symbolic numbers.

Much, however remains unclear about the role that the ANS plays in grounding numerical symbols. First, the process by which the quantities represented by the ANS are mapped onto symbolic numbers is not at all clear. Moreover, it is not clear that there are not other equally important pathways by which humans might gain access to semantics of numerical magnitude in order to ground numerical symbols. For instance, some have argued that magnitude comparison of numerical and non-numerical stimuli is accomplished using similar mechanisms (Walsh 2003). This suggests the possibility that some non-ANS based mode of expressing magnitude – such as the visuospatial representation of relative length – might provide an alternative ground making symbolic numbers meaningful. In fact, space is a particularly strong candidate for grounding symbolic number, as several lines of research show strong associations between number and space (see Hubbard Piazza, Pinel & DeHaene, 2005 for a review). This association between magnitude and space is strongly implicated in the second type of number sense, discussed below.

Sense 2 – Number Line Estimation

Siegler, Booth and colleagues have taken a somewhat different tack in exploring number sense. These researchers have defined number sense as the ability to estimate numerical magnitudes. this take on the number sense focuses on the ability to generate numbers whose magnitudes are close to correct values for the various types of estimation questions posed. Perhaps the most well known of these numerical estimation tasks is the number line estimation task. This task requires that participants estimate the location of a number on a line with numerical anchors at each end. This task not only involves magnitude estimation, but also the ability to translate between symbolic numbers and mental magnitudes.

Siegler and colleagues have helped illuminate a typical developmental trend observed on such number line tasks: young children show a tendency to compress numbers logarithmically, whereas adults do not (e.g. Siegler & Booth, 2004; Booth & Siegler, 2006; Booth and Siegler, 2008. The figure below is from Siegler & Booth, 2004). That is, when asked to mark the position where a number n is on the line, children typically place the mark at a spot approximated by $\log(n)$ instead of n. With development, these estimates become more linear (Siegler & Opfer, 2003). Importantly, increased linearity of children's magnitude estimates predicts a wide range of numeracy measures, including counting ability, number naming, digit magnitude comparison, and achievement test scores (Ramani & Siegler, 2008; Siegler & Booth, 2004).

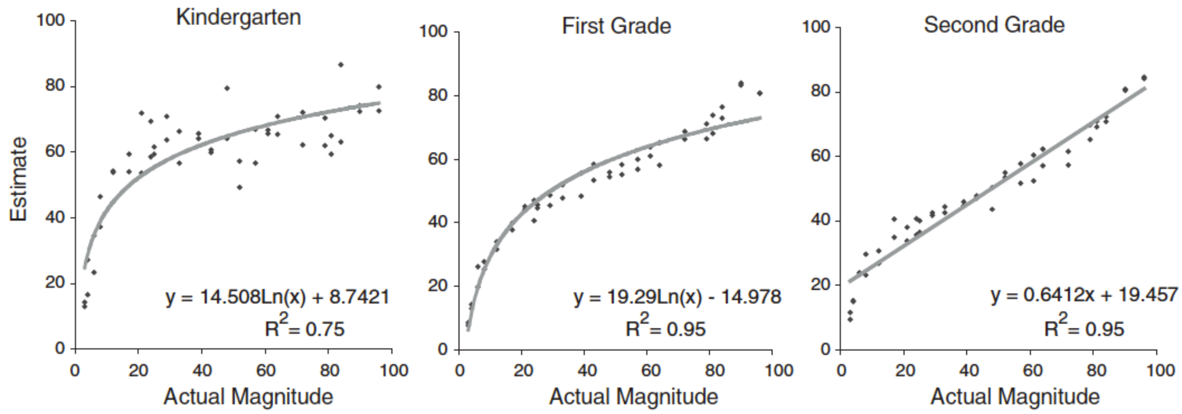

Figure 2. Progression from logarithmic pattern of median estimates among kindergartners (left panel) to linear pattern of estimates among second graders (right panel) in Experiment.

This developmental trend, labeled the 'log-to-linear shift', has been interpreted as evidence of a developmental change in the underlying subjective representations of the numerical magnitudes denoted by symbolic numbers. According to the log-to-linear shift hypothesis, the curve found in young children's' number line estimations is analogous to the curvature found in other raw estimation tasks (as described above). This interpretation presupposes that children's performance on number line estimation tasks initially reflects the perceptual influence of the approximate number system that we share with nonhuman animals (Siegler, 2009; Siegler & Opfer, 2003). Siegler and colleagues further theorize that when children start to make linear predictions, they are moving away from reliance on the core perceptual system of the ANS to reliance on the formal, school-taught system. This explanation of the log-to-linear shift is currently popular in cognitive psychology (e.g. Dehaene, Izard, Spelke & Pica, 2008), developmental psychology (e.g. Whyte & Bull, 2008), and special education research (e.g., Fuchs, Geary, Compton, Fuchs, Hamlet & Bryant 2010). It has even influenced the policy oriented reports of the National Math Advisory Panel (2008).

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III. A CRITIQUE OF THE LOG-TO-LINEAR SHIFT HYPOTHESIS

It is important that I offer a critique of the log-to-linear shift hypothesis, because the paradigm is somewhat contradictory to the proportion matching account that I offer. To summarize the discrepancy, my proportion matching account relies upon the premise that using the perceptual system to make estimations on bounded lines (like number lines, and other perceptually-based tasks that are similar to number lines, see figure below) should yield linear estimation patterns. The log-to-linear shift hypothesis, however, predicts that using the perceptual system should lead to logarithmically curved estimates on these tasks.

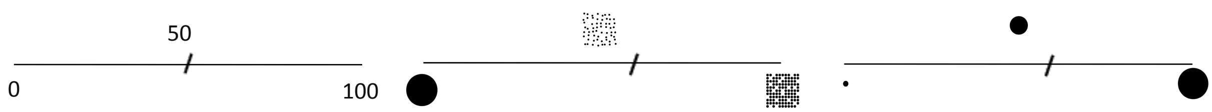


FIGURE: Each of these three tasks requires estimating where a stimulus (above the line) should be place on a line bounded by stimuli at both ends. Each requires matching proportions of presented stimuli to the corresponding proportions of line lengths (for more on this, see Matthews & Chesney, 2011).

My critique relies heavily upon that offered by Barth and Paladino (2010). I extend the argument by providing direct evidence that viewing magnitudes in isolation vs. viewing those same magnitudes on bounded lines should automatically result in linear estimates, even when the perceptual system is used to judge magnitudes. I also extend the argument of Barth and Paladino by showing that proportions between magnitudes in multiple formats (not just line lengths) can be mapped back to symbolic numbers. The section below will provide a bit more information on why number line estimation should be seen as a type of proportion judgment.

The log-to-linear shift hypothesis is not without controversy. A recent critique put forth by Barth and Paladino (2010) raised questions about the interpretation of children's apparently logarithmic performance (see also Cohen & Blanc-Goldhammer, 2011). Barth and Paladino argue that estimating the proper placement of a number on a number line is not quite the same as estimating the size of a number seen in isolation. Rather, such placement tasks are actually a form of proportion judgment task – a task in which the ratio between items must be evaluated. Indeed, previous literature in psychophysics has shown that estimation tasks that combine two measures in a complementary fashion such that they sum to a fixed total should be characterized as proportion judgment tasks (e.g., Hollands & Dyre, 2000; Spence, 1990; Stevens, 1957). Thus, because estimating a number's place on a number line involves both the estimate of that numbers' placement relative to the zero anchor point and of its complement's placement relative to the rightmost anchor, the task is essentially a proportion judgment. For example, when placing 25 on a 0-100 line, it should be 25 units away from 0, and 75 units away from 100, and should therefore be placed 25/(25+75), or one fourth of the total length of the line away from 0.

Proportion judgment tasks tend to yield linear relationships between the actual proportion of the stimulus presented and the judged proportion, even when using stimuli for which estimation in isolation yields a curved relationship between actual and perceived stimulus intensity (Hollands & Dyre, 2000; Spence, 1990). This linear performance results because the underlying curved function is mapped to distances according to what Spence (1990) termed a power model, which approximates linearity because of the reference points that are perceived linearly. Importantly, this model predicts linear performance even given a curved underlying representation of number (see Figure below, reproduced from Hollands & Dyre, 2000).

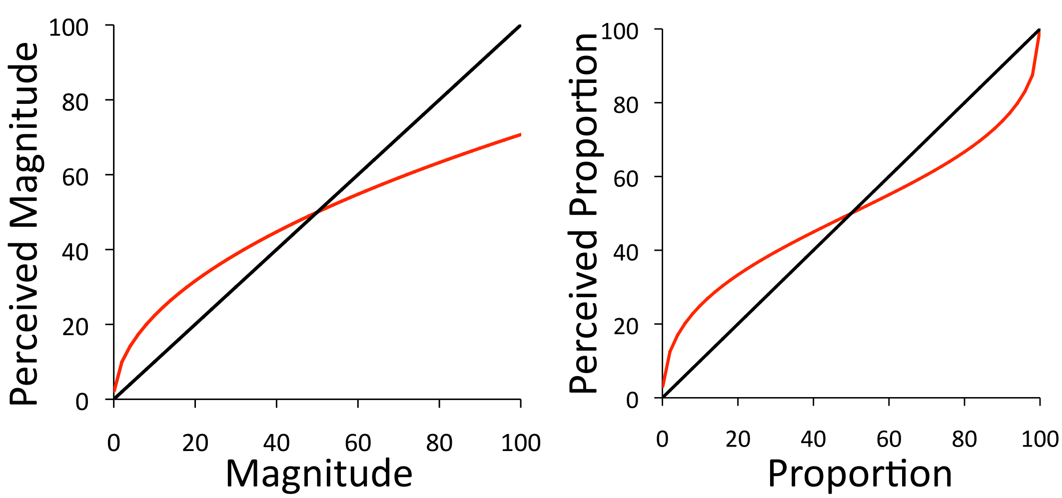
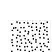
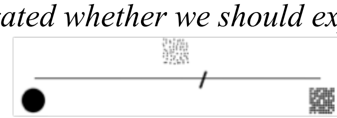


FIGURE: According to psychophysics models, seeing a magnitude in isolation should yield a function like the one on the left, whereas judging the magnitude as a proportion of whole should yield a function on the right. As an application, I've investigated whether we should expect different patterns when investigating stimuli such as  *vs.* .

This perspective raises questions regarding young children's apparently logarithmic performance on number line estimation tasks. Why, given that number line estimation is a proportion judgment task, do children's representations appear to be nonlinear in the first place, when psychophysics – even given a logarithmic underlying representation of number magnitude – might predict otherwise? One possible answer is that certain assumptions of the psychophysics proportion judgment model may be violated when young children perform number line estimation tasks, impeding their use of the default comparison procedure for performing the tasks.

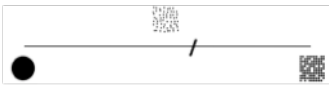
To judge a proportion, one must know the approximate magnitude of the whole (i.e., the rightmost anchor on a number line). Indeed, the proportion judgment model assumes that participants have access to the magnitudes at both ends of the line. Although this assumption is logical when perceptual continua are used to indicate the anchors at each end of the line (e.g., bar length on a bar graph, see Spence, 1990), this is not necessarily the case with tasks that require young children to understand the magnitudes of symbolic numbers. Essentially, people who do not have a correct understanding of the values represented by both the high and low anchor points lack the knowledge needed to fully render number line estimation tasks as proportion judgment tasks.

Extant research on children's abilities to identify symbolic numbers by name provides some evidence children do lack the symbol knowledge that is assumed by current interpretations of number line estimation patterns. For example, young children often cannot consistently name symbolic numbers above twenty, even when they can recite those numbers as part of the count sequence (Wright, 1991; see also Clarke & Shinn, 2004). One might question whether estimates based on any unrecognizable number should have a one-to-one mapping to any particular numerical magnitude. Unfortunately, no number line estimation experiments have included alternative measures of children's knowledge of number symbols for symbols greater than 20.

I will present data from an experiment with Kindergarten participants that includes such alternative measures alongside number line estimation data. The evidence is consistent with my claim that these children's ostensibly logarithmic performance on 0-100 number lines may actually be an artifact of children's unfamiliarity with number symbols. I will show that children's knowledge of number symbols in this case was insufficiently stable to support the log-to-linear shift hypothesis. These arguments have important implications for the validity of what is currently a very popular methodology.

IV. A SENSE OF PROPORTION

Much of the literature on the perception of magnitude argues that our perception of magnitude is sensitive to ratio. To date, this ratio sensitivity has been deduced either 1) from reaction times in discrimination tasks or 2) from the characteristic curvature found in magnitude estimation tasks. My findings motivated me to investigate whether college undergraduates could directly generate fractions corresponding to the ratios they perceived in figures like the ones shown below. I presented figures like these too rapidly for participants to count or to use equations (1500 ms), instructing them instead to feel out the ratios.

Despite their lack of confidence, undergraduates were amazingly accurate at assigning appropriate fraction values to these visuospatially presented ratios. This means that we have direct perceptual access to the ratios between pairs of magnitudes. Moreover, variance patterns for these tasks were similar to those obtained from the analogs to the number line estimation (e.g.  *). This can be interpreted as further confirmation that tasks requiring participants to estimate the placement of magnitudes on bounded lines (including number line estimation) are indeed proportion matching tasks. Unpacking the cross-representational proportion matching aspect of the task opens up the possibility that the one mechanism by which number lines help improve symbolic number knowledge may involve leveraging our perceptual system to inform our understandings of the relative magnitudes between symbolic numbers.*

As you'll note, this section doesn't include citations. This is simply because there aren't any. The paradigm is original,[4] *and I'm in the process of fleshing it out and exploring its potential applications for pedagogical purposes.*

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[4] Actually, I did find a reference to some sort of proportion judgment task in Taves (1941) that may possibly have examined a similar phenomenon. The problem is that there were no figures, and textual explanation of the design was too brief to get a picture of what all was involved with the task.

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